9. Problemset "Theoretical Particle Physics"
June 21, 2017

Custodial Symmetry

9.1 so(4) ≅ su(2) × su(2)

1. Find a basis for the Lie algebra so(4) generating the group SO(4) of orthogonal 4 × 4-matrices with unit determinant.

2. Show that generators of so(4) can be expressed by linear combinations of generators of two copies of su(2). In other words, show that so(4) ≅ su(2) × su(2).

9.2 SO(4) vs. SU(2) × SU(2)

Just as so(3) ≅ su(2) does not imply that the groups are isomorphic (just SO(3) ≅ SU(2)/$\mathbf{Z}_2$), we can *not* expect SO(4) ≅ SU(2) × SU(2).

1. Find the exact relationship of SO(4) and SU(2) × SU(2).

9.3 SO(4) vs. SU(2) × SU(2) redux

Given a set X and a group G, we say that G acts on X from the left, if there is a map

$$\begin{aligned} \triangleright : G \times X &\to X \\ (g, x) &\mapsto g \triangleright x \end{aligned} \tag{1}$$

and from the right

$$\begin{aligned} \triangleleft : G \times X &\to X \\ (g, x) &\mapsto x \triangleleft g \,, \end{aligned} \tag{2}$$

iff these actions are compatible with the group structure

$$(gg') \triangleright x = g \triangleright (g' \triangleright x) \tag{3}$$

and

$$(gg') \triangleleft x = (x \triangleleft g') \triangleleft g \tag{4}$$

etc.

1. Find canonical left and right actions for the special case $X = G$.

2. Show that

$$\begin{aligned} \blacktriangleright \colon (G \times H) \times X &\to X \\ ((g,h),x) &\mapsto (g \times h) \blacktriangleright x = g \triangleright x \triangleleft h \end{aligned} \tag{5}$$

defines a left action of $G \times G$ on G. Does this construction also work for an action of $G \times H$ on X?

3. Find a parameterization of the matrices in the set SU(2) by $\alpha \in \mathbf{R}^4$ with $\alpha^2 = 1$. Use it to show that the action

$$(\mathrm{SU}(2) \times \mathrm{SU}(2)) \blacktriangleright \mathrm{SU}(2)\,. \tag{6}$$

can also be understood as an action

$$\mathrm{SO}(4) \blacktriangleright \mathrm{SU}(2)\,. \tag{7}$$

What does this mean for the relationship of SO(4) and $\mathrm{SU}(2) \times \mathrm{SU}(2)$?