1. Problemset "Theoretical Particle Physics"
April 24, 2017

Pauli Matrices &c.

Since we will have to deal a lot with small matrices, it's a good idea to review some useful formulae for concrete calculations. Particularly important are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

which form a basis for the traceless and hermitian 2×2-matrices and satisfy

$$\sigma_i \sigma_j = \delta_{ij} \mathbf{1} + \mathrm{i} \sum_{k=1}^{3} \epsilon_{ijk} \sigma_k \,. \tag{2}$$

1.1 Exponentials

Use the Pauli matrices to parametrize a general complex 2×2-matrix M by four complex numbers $(a_0, \vec{a})$

$$M(a_0, \vec{a}) = a_0 \mathbf{1} + \vec{a}\vec{\sigma} = a_0 \mathbf{1} + a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3 = \begin{pmatrix} a_0 + a_3 & a_1 - \mathrm{i} a_2 \\ a_1 + \mathrm{i} a_2 & a_0 - a_3 \end{pmatrix} \tag{3}$$

and show that

$$\exp\left(\mathrm{i} M(a_0, \vec{a})\right) = \mathrm{e}^{\mathrm{i} a_0} M\left(\cos a, \mathrm{i} \frac{\sin a}{a} \vec{a}\right) \qquad \text{with} \quad a = \sqrt{\vec{a}^2} \,. \tag{4}$$

1.2 Complex Conjugate Representation

Consider the matrices

$$\tilde{\sigma}_i = -\sigma_2 {\sigma_i}^* \sigma_2 \qquad (\text{for } i = 1, 2, 3) \tag{5}$$

where $\cdot^*$ denotes elementwise complex conjugation and *not* hermitian conjugation.

Compute their commutation relations

$$[\tilde{\sigma}_i, \tilde{\sigma}_j] \,. \tag{6}$$

1.3 Hausdorff's Formula

Prove Hausdorff's formula

$$\mathrm{e}^{A} B \mathrm{e}^{-A} = \mathrm{e}^{\mathrm{ad}_A} B \tag{7}$$

with the adjunction operator ad_A

$$\mathrm{ad}_A B = [A, B] \tag{8}$$

for arbitrary operators and matrices.

Hint: replace $A \to tA$ (with $t \in \mathbf{R}$) on both sides and show, that both sides of the equation solve the same initial value problem.