4. Problemset "Quantum Algebra & Dynamics"
November 6, 2015

More Functions / Subalgebras

4.1 Square Root and Exponential

Use again the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

to parametrize a general complex 2×2-matrix $M \in \mathcal{M}_2$ by four complex numbers $(a_0, \vec{a})$

$$M(a_0, \vec{a}) = a_0 \mathbf{1} + \vec{a}\vec{\sigma}\,. \tag{2}$$

1. Use the holomorphic functional calculus to compute $\sqrt{M}$.

2. Use the holomorphic functional calculus to compute $\mathrm{e}^{\mathrm{i}M}$ and compare the result with the corresponding power series.

4.2 Square Root Revisited

Show $B^2 = A \geq 0$ for

$$B = \int_0^\infty \frac{\mathrm{d}x}{\pi} \frac{A}{\sqrt{\lambda}} \frac{1}{\lambda \mathbf{1} + A}\,. \tag{3}$$

by explicit calculation without using the holomorphic functional calculus.

4.3 Subalgebras

Consider a C^*-algebra $\mathcal{A}$, an element $P = P^* \in \mathcal{A}$ with

$$P^2 = P\,. \tag{4}$$

Show that the subset

$$\mathcal{A}' = P\mathcal{A}P = \{PAP : A \in \mathcal{A}\} \subseteq \mathcal{A} \tag{5}$$

is a C^*-algebra with identity $\mathbf{1}_{\mathcal{A}'} = P$.