Entanglement entropy, Krylov complexity and Deep inelastic scattering data

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Krzysztof Kutak

Based on:

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Eur.Phys.J.C 82 (2022) 2, 111
M. Hentschinski, K. Kutak

Eur.Phys.J.C 82 (2022) 12, 1147
M. Hentschinski, K. Kutak, R. Straka

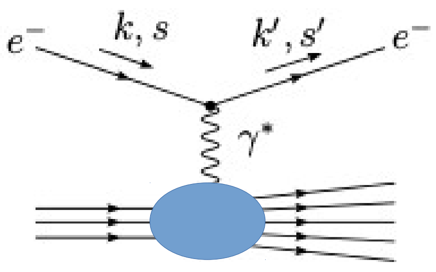
PRL'23
H. Hentschinski, D. Kharzeev. K. Kutak, Z. Tu

Rept.Prog.Phys. 87 (2024) 12, 120501
M. Hentschinski, D. Kharzeev, K. Kutak, Z. Tu

Phys.Rev.D 110 (2024) 8, 085011
P. Caputa, K. Kutak

Deep Inelastic Scattering

$$p_e = E_e(1,0,0,1)$$



$$p_p = E_p(1,0,0,-1)$$

Parameters from DESY

$$E_{e^-} \sim 27 \text{ GeV}$$

interesting in the context of
future experiment
Electron Ion Collider at BNL and
Forward Physics Facility at CERN

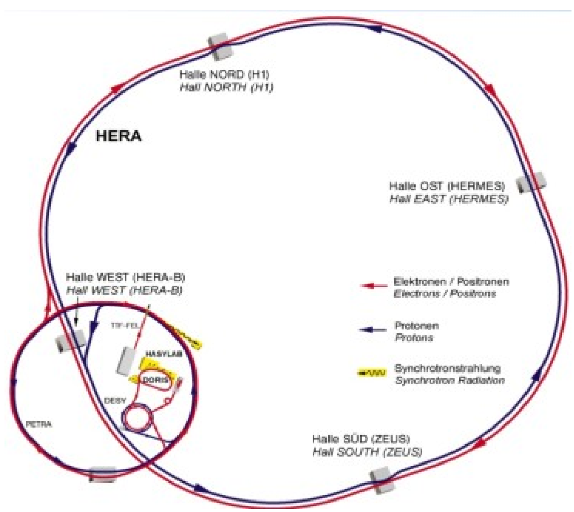
$$E_P \sim 820 \text{ GeV}$$

$$\sqrt{s} \sim 300 \text{ GeV}$$

$$E_\gamma \sim \sqrt{Q^2} \sim k$$

From the uncertainity principle we get $\Delta x \, \Delta k \sim 1$

$$\Delta x \sim \frac{1}{k} \sim \frac{1}{\sqrt{Q^2}}$$



The final state depends on the virtuality of photon $Q^2 = -q^2$

DIS reactions are going to be studied at

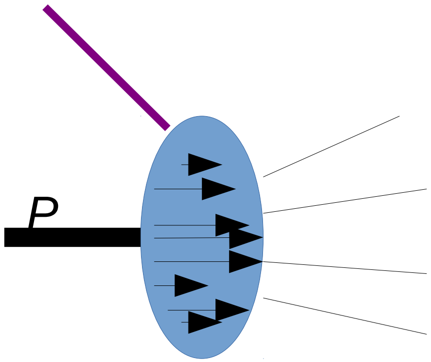
Electron Ion Collider, USA ~ 2030

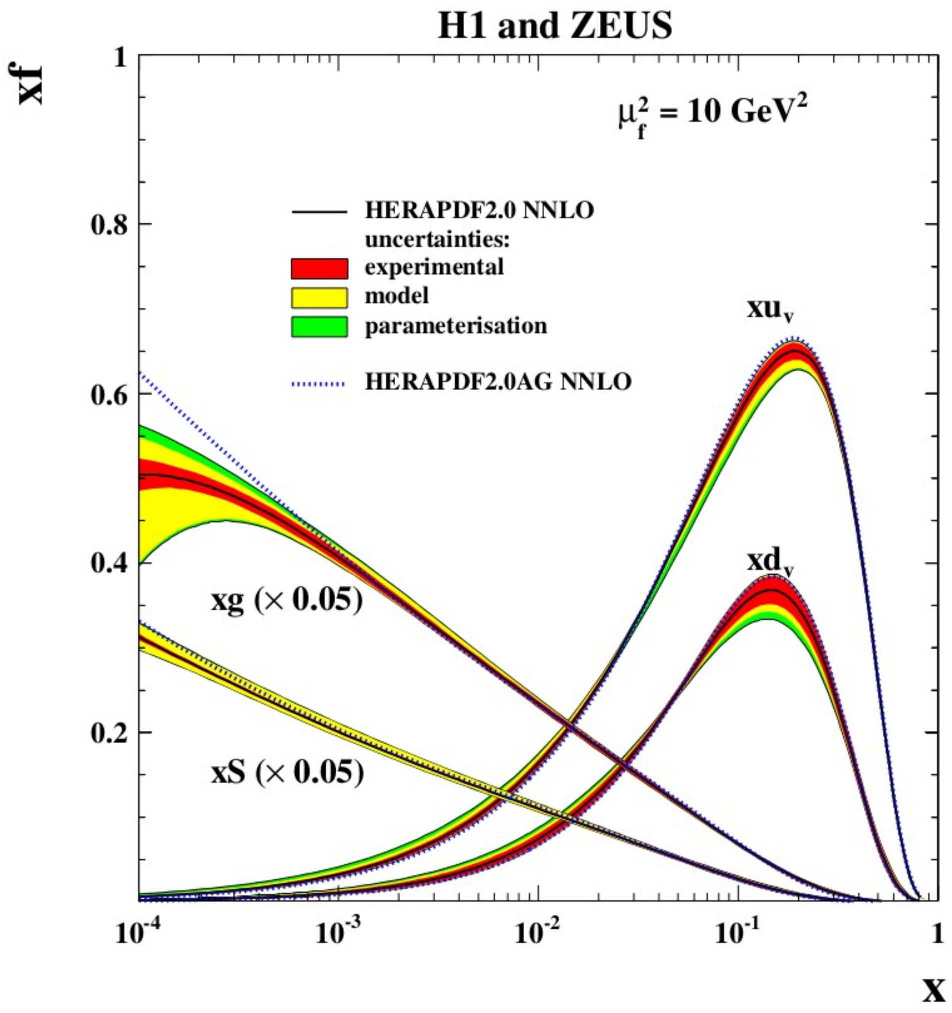
Forward Physics Facility at CERN

Structure of the proton – collinear factorization

$$k = xP$$

$$g(x, Q^2) \sim \alpha_s \ln \frac{Q^2}{Q_0^2}$$





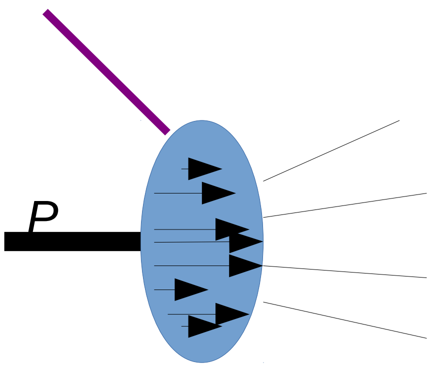
The collinear pdfs are known up to NNLO.

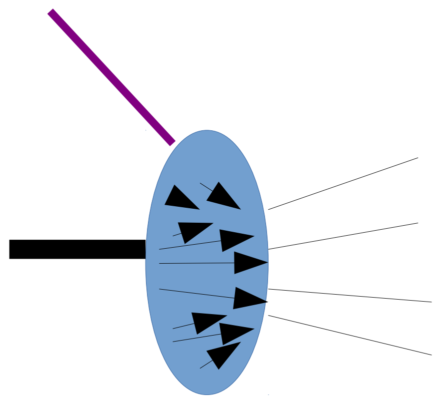
Collinear factorization most efficiently used for central production with large scales.
Many processes known at high accuracy

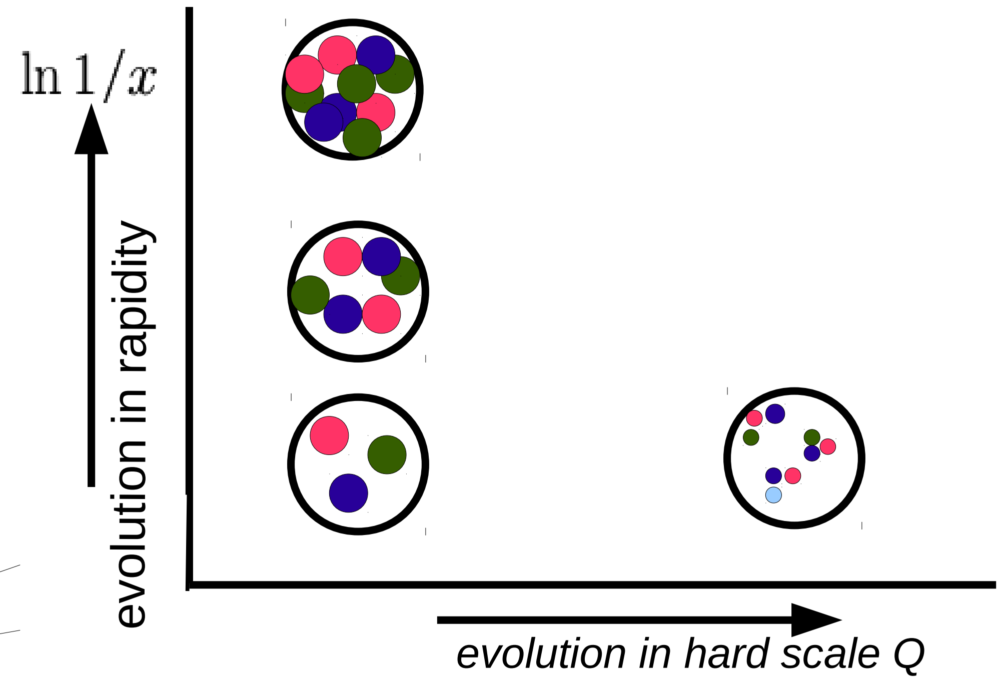
Collinear vs. k_t factorization - kinematics

$$k = xP \qquad \text{vs.} \qquad k = xP + k_T$$

$$g(x,Q^2) \sim \alpha_s \ln \frac{Q^2}{Q_0^2} \qquad\qquad g(x,Q^2) \sim \alpha_s \ln \frac{x_0}{x}$$



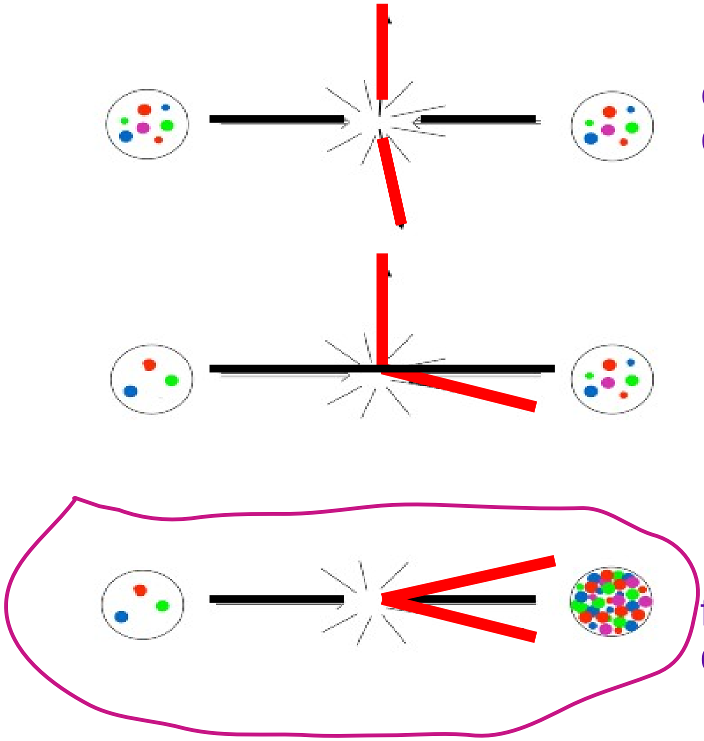




In the whole discussion I sum over helicities.
i.e. I consider spin independent pdfs.

Many processes in the k_t factorization are known at NLO
However the general method and Monte Carlo
Implementation is in a process of developmenet

The LHC context – dijet production



central-central i.e.
dilute dilute

$$x \sim 10^{-1}$$

forward-central

$$x \sim 10^{-4}$$

forward-forward i.e.
dilute - dense

$$x \sim 10^{-5}$$

Measurements planned within ALICE FoCal detector – 2030 and possibly with upgrades of ATLAS

Review:
Searching for saturation in forward dijet production at the LHC '23
K. Kakkad, P. Kotko, K. Kutak, S. Sapeta, A. Van Hameren

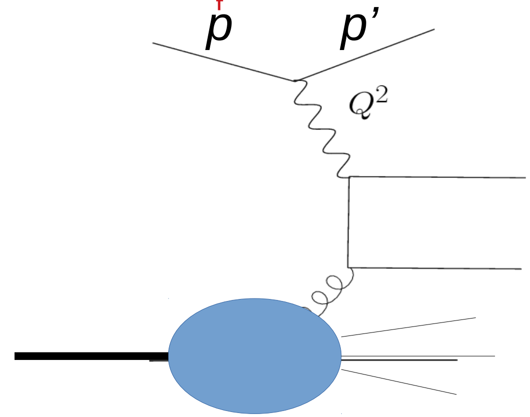
DIS proton structure function and dipole cross section

$$F_2(x,Q^2) = \frac{Q^2}{4\pi^2}\alpha_s \sum_q e_q^2 \int d^2k\, \mathcal{F}(x,k^2) \left(S_L(k^2,Q^2,m_q^2) + S_T(k^2,Q^2,m_q^2)\right)$$

dipole gluon density

impact factors ~ hard coefficients

In the k_t factorization

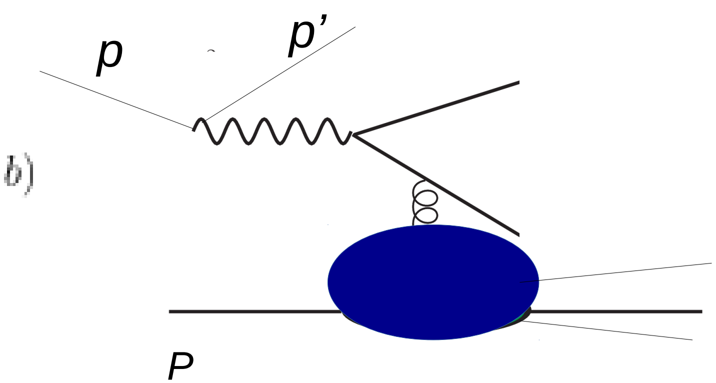


boosted proton

$$xg(x,Q) = \int_0^{Q^2} d\mathbf{k}^2 \mathcal{F}(x,\mathbf{k}^2)$$

number of gluons as “seen” at scale Q

In the dipole formalism



proton in rest frame

$$F_2(x,Q^2) = \frac{Q^2}{4\pi^2\alpha_{em}} \int d^2b \int_0^1 dz \int d^2r \left(|\psi_L(z,r)|^2 + |\psi_T(z,r)|^2\right) N(x,r,b)$$

wave function

dipole amplitude

Gluon distribution

BFKL gluon distribution

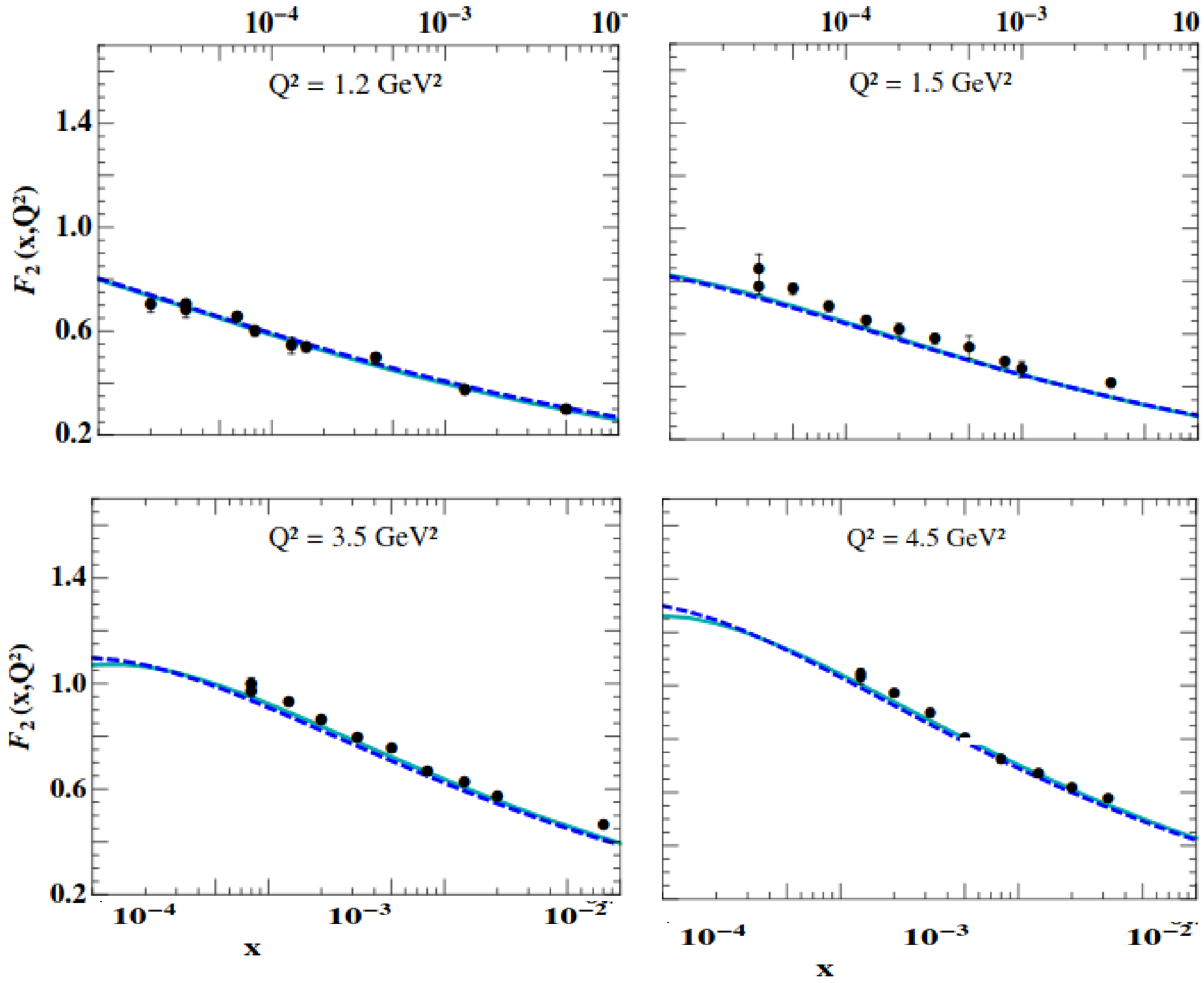
$$\mathcal{F}\left(x, \boldsymbol{k}^2, Q\right) = \frac{1}{\boldsymbol{k}^2} \int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\gamma}{2\pi i}\ \hat{g}\left(x, \frac{Q^2}{Q_0^2}, \gamma\right)\ \left(\frac{\boldsymbol{k}^2}{Q_0^2}\right)^{\gamma}$$

$$\hat{g}\left(x, \frac{Q^2}{Q_0^2}\gamma\right) = \frac{\mathcal{C}\cdot\Gamma(\delta-\gamma)}{\pi\Gamma(\delta)} \cdot \left(\frac{1}{x}\right)^{\chi(\gamma,Q,Q)} \left\{1 + \frac{\bar{\alpha}_s^2 \beta_0 \chi_0\left(\gamma\right)}{8N_c} \log\left(\frac{1}{x}\right)\left[-\psi\left(\delta-\gamma\right) + \log\frac{Q^2}{Q_0^2} - \partial_\gamma\right]\right\}$$

the low x growth

Proton structure function

F_2 data description



The motivation

Properties of entanglement entropy may provide some new insight on understanding of behavior of parton density functions

Links to other areas (thermodynamics, gravity, quantum information, conformal field theory)

Interesting in context of parton saturation and thermalization problem of Quark Gluon Plasma

Recent progress in the field comes from applying these ideas in the context of Deep Inelastic scattering

Entropy in stat. mech. – reminder

In statistical physics the entropy S of macrostate is given by the *log* of number *W* of distinct microstates that compose it

$$S = -\sum_{i=1}^{W} p(i) \ln p(i)$$

Gibbs entropy

Boltzmann entropy

For uniform distribution $p(i) = \frac{1}{W}$ the entropy is maximal

$$S = \ln W$$

If probability of state is 1 entropy is 0. Entropy in the information sense theory tells us about the amount of missing information.

Since partons are introduced as the microscopic constituents that compose the macroscopic state of the proton, it is natural to evaluate the corresponding entropy or entropy corresponding to parton density.

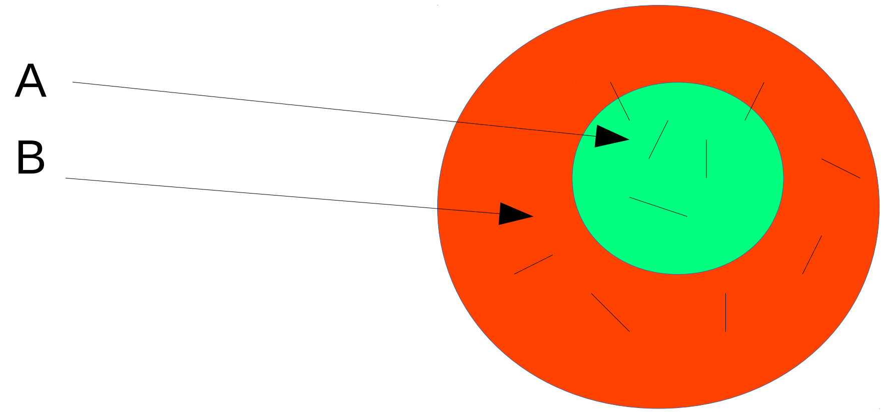
K. Kutak '11, Peschanski '12

I. Zahed, A. Stoffers '13

A. Kovner, M. Lublinsky '15, ...

Entanglement and entropy

$\mathcal{H}_A \otimes \mathcal{H}_B$



The composite system is described by

$|\Psi_{AB}\rangle$ in $A \cap B$

general definitions

entangled
if the product can not be expressed as separable product state

$$|\Psi_{AB}\rangle = \sum_{i,j} c_{ij} |\varphi_i^A\rangle \otimes |\varphi_j^B\rangle$$

separable
if the product can be expressed as separable product state

$$|\Psi_{AB}\rangle = |\varphi^A\rangle \otimes |\varphi^B\rangle$$

$\mathcal{H}_B$ of dimension n_B.

$\mathcal{H}_A$ of dimension n_A

Schmidt decomposition

$$|\Psi_{AB}\rangle = \sum \alpha_n |\Psi_n^A\rangle |\Psi_n^B\rangle$$

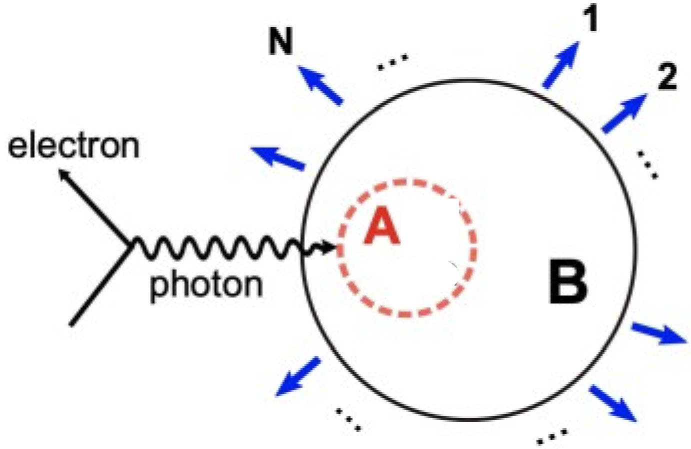
orthonormal states belonging to A

orthonormal states belonging to B

related to matrix C

Entanglement and entropy and DIS

$$\mathcal{H}_A \otimes \mathcal{H}_B$$



$$|\Psi_{AB}\rangle = \sum_n \alpha_n |\Psi_n^A\rangle |\Psi_n^B\rangle$$

$$\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$$

$$\rho_A = \mathrm{tr}_B \rho_{AB} = \sum_n \alpha_n^2 |\Psi_n^A\rangle\langle\Psi_n^A|$$

The density matrix of the mixed state probed in “region” A.

$$S_A = -\rho_A \ln(\rho_A) = S_B$$

Resolution in the longitudinal degrees of freedom

$$S_A = -\sum_n p_n \ln p_n \qquad \alpha_n^2 \equiv p_n$$

Kharzeev, Levin ‘ 17

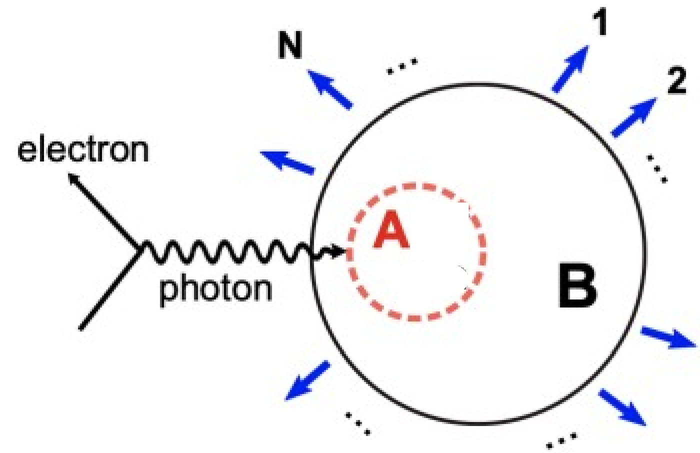
Shannon entropy i.e. the information content, of the probability distribution

Analogy

black hole + coffee → Hawking radiation + back hole

electron + proton → electron +radiation of hadrons

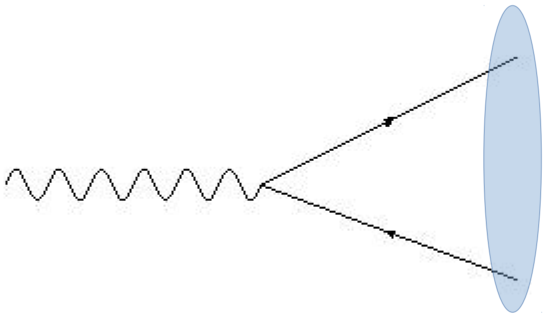
J. A. Wheller thought experiment.

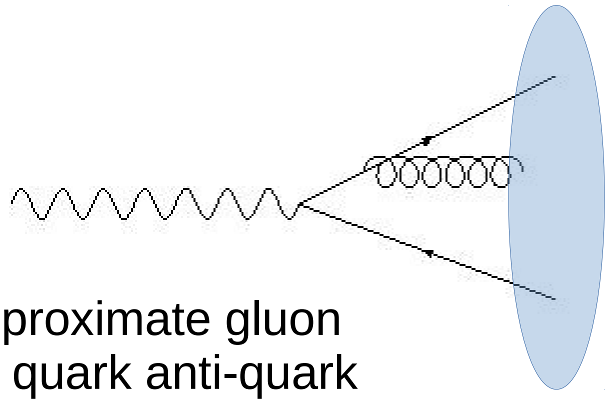


Bekenstein: BH has to have entropy because coffee has entropy and overall entropy would decrease if we toss coffee into black hole without increasing entropy

Comment:
better analogy would be proton – ion collision

Cascade of dipoles

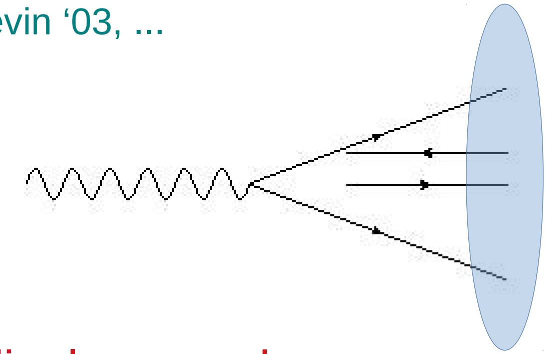


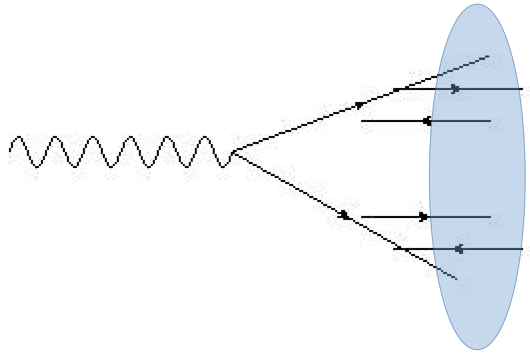


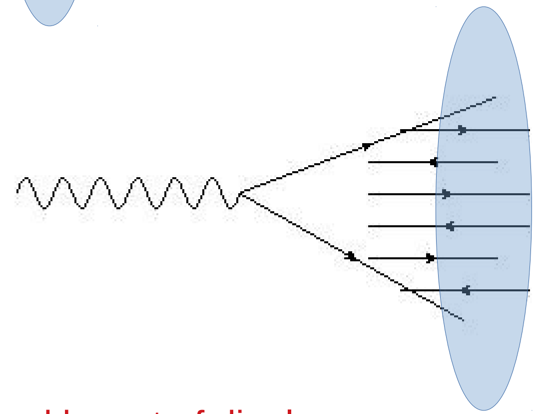
approximate gluon
by quark anti-quark

Microstates p_n – probabilities for given number of dipoles

Mueller 95,
Lublinsky, Levin ‘03, ...





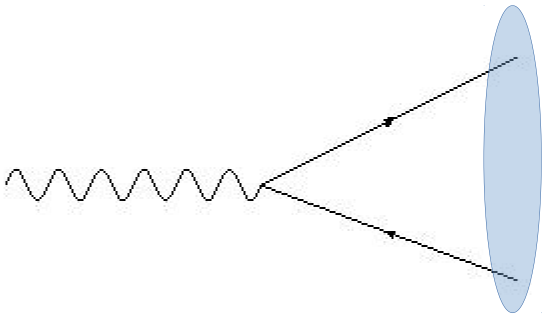


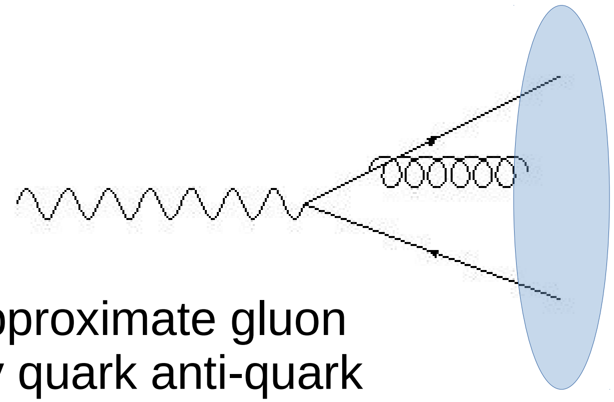
BFKL dipole cascade

set of partons is described by set of dipoles with fixed sizes ,Y is rapidity and is related to energy

- One considers state of $\bar{q}$ q and successive emission of gluons.
- Tracing over coordinates of large x modes will give reduced will density matrix of soft gluons from which one can the calculate entropy.
- Explicit construction in Liu Nowak, Zahed ‘22

Cascade of dipoles – fixed dipole size





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Mueller 95,
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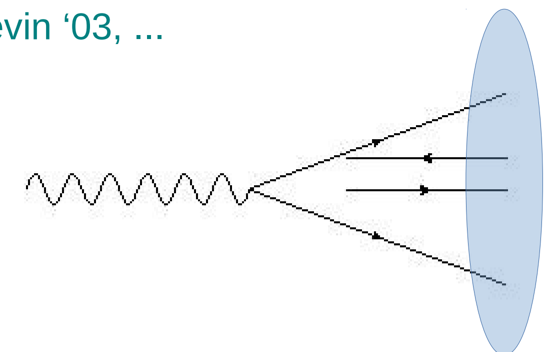
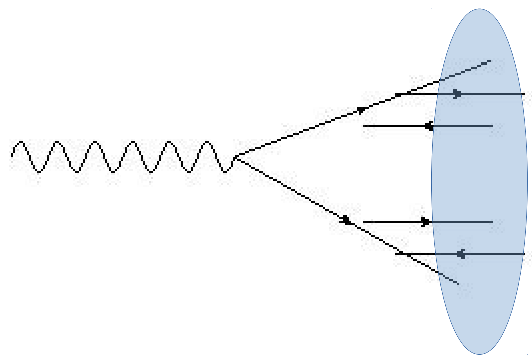
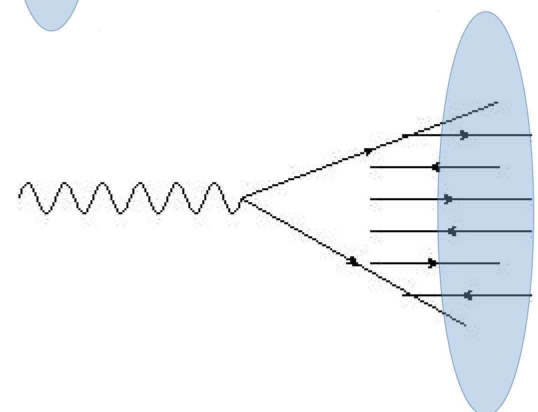






BFKL dipole cascade

set of partons is described by set of dipoles with fixed sizes ,Y is rapidity and is related to energy

$$|\Psi_n\rangle = \sum_{x_0 \gg x_1 \gg x_2 \ldots \gg x_n \gg x_{\min}} \sqrt{a^n n!}\frac{e^{-\frac{ay+a\sum_{i=1}^{n} y_i}{2}}}{\sqrt{(\Lambda^-)^n x_1 .. x_n}} |x_1\rangle |x_2\rangle ..... |x_n\rangle$$

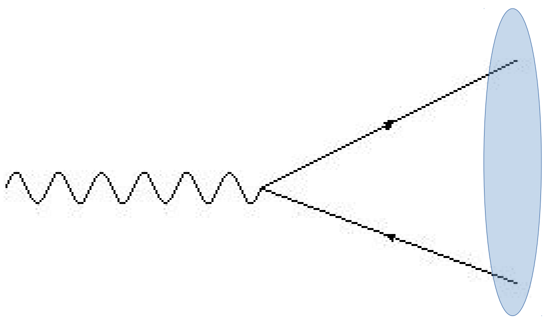
Liu, Nowak, Zahed ‘22

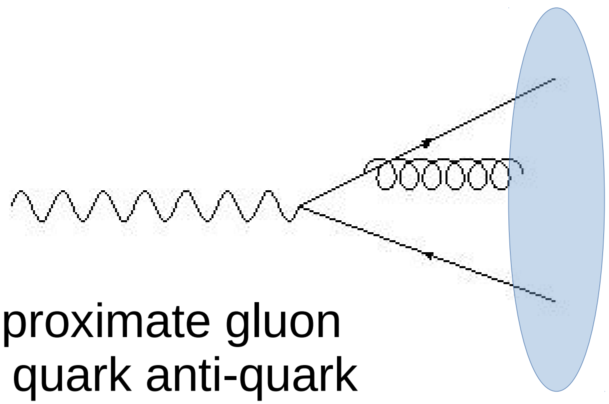
$$\hat{\rho}_1 = \sum_{n=0}^{\infty} |\Psi_n\rangle\langle\Psi_n|$$

State of n dipoles with
longitudinal momenta

$$y_i = \ln \frac{x_i}{x_{\min}}$$

Cascade of dipoles





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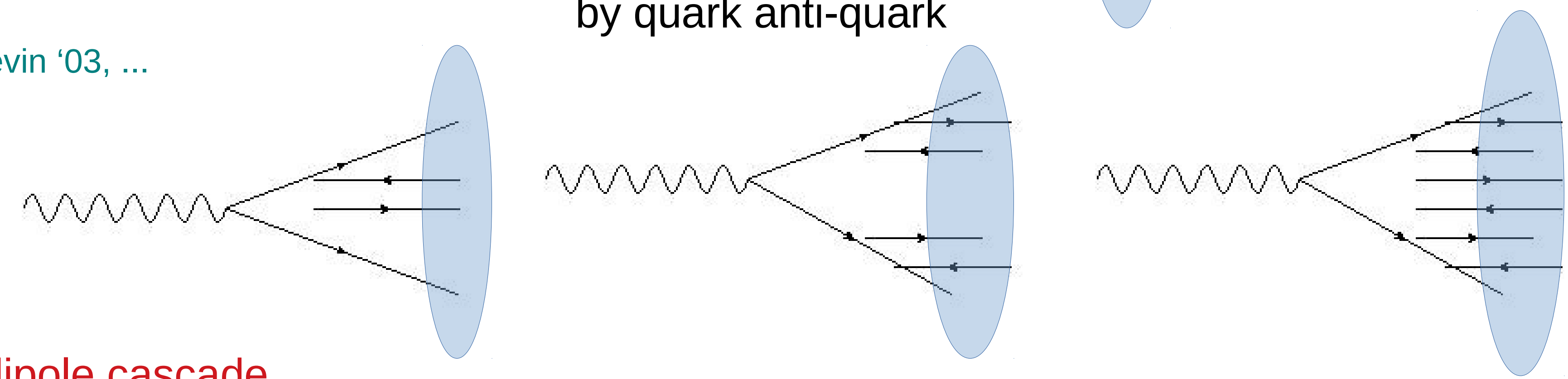


BFKL dipole cascade

$$\partial_y p_n(y,\{r\}) = \sum_m K \otimes p_m(y,\{r\})$$

probability to find n dipoles an rapidity y

transverse sizes of dipoles

set of partons is described by set of dipoles with fixed sizes ,Y is rapidity and is related to energy

$$P_n(Y) = \frac{1}{n!}\int d^2r_1 d^2b_1 \cdots d^2r_n d^2b_n \frac{|\Psi^{[n]}(\vec{r}_{1\perp}, \vec{b}_{1\perp}, \ldots, \vec{r}_{n\perp}, \vec{b}_{n\perp}, Y)|^2}{\sum_{\sigma\sigma'} |\Psi^{(0)}_{\sigma\sigma'}(\vec{x}_{10}, z_1)|^2 \Big|_{O(\alpha_s^0)}},$$

$$Y \equiv y$$

Cascade of dipoles – fixed dipole size

Initial conditions

$$\frac{dp_n(y)}{dy} = -\lambda\, n\, p_n(y) + (n-1)\, \lambda\, p_{n-1}(y)$$

$p_1(0) = 1$ at initial rapidity there is only 1 dipole

$p_{n>1}(0) = 0$

rate at which number of dipoles grow. The phenomenological value is $\lambda = 0.3$.
It is an observable.

depletion of the probability to find n dipoles due to the splitting into (n + 1) dipoles.

the growth due to the splitting of (n − 1) dipoles into n dipoles.

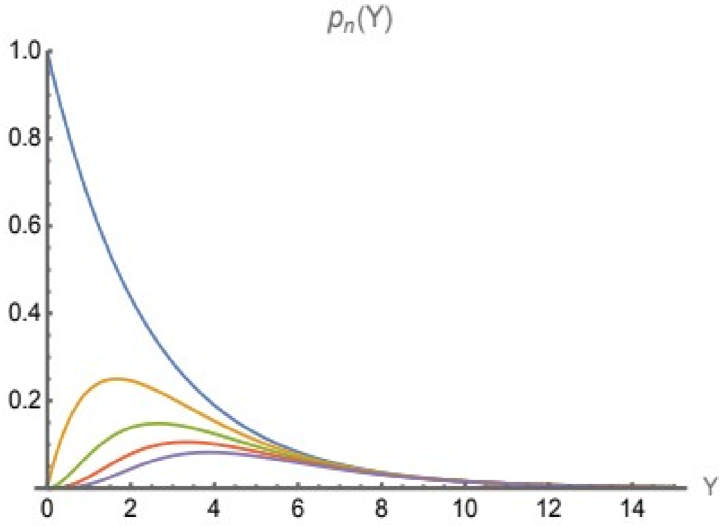
See for density matrix and 3+1 dimensional case in DLL and KNO function
Liu, Nowak, Zahed '22 PRD

Exact analytical solution is:

$$p_n(y) = e^{-\lambda y}(1 - e^{-\lambda y})^{n-1} \qquad y = \ln\left(\frac{1}{x}\right)$$

Cascade of dipoles – fixed dipole size

$$\frac{dp_n(y)}{dy} = -\lambda\, n\, p_n(y) + (n-1)\, \lambda\, p_{n-1}(y)$$



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KL entropy formula - interpretation

$$p_n(y) = e^{-\lambda y}(1 - e^{-\lambda y})^{n-1}$$

At low x partonic microstates have equal probabilities.

In this equipartitioned state the entropy is maximal – the partonic state at small *x* is maximally entangled.

In terms of information theory as Shanon entropy:

- equipartitioning in the maximally entangled state means that all “signals” with different number of partons are equally likely

- it is impossible to predict how many partons will be detected in a give event.

- structure function at small x should become universal for all hadrons.

Kharzeev, Levin ‘ 17

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Cascade of dipoles – entropy

Kharzeev, Levin ' 17

$$\langle n \rangle_y = \sum_n n p_n(y) \equiv \bar{n}(x)$$

mean number of dipoles / partons

Using the p_n from previous slide we get

$$\langle n \rangle_y = \sum_n n p_n(y) \equiv \bar{n}(x) = \left(\frac{1}{x}\right)^{\lambda}$$

Gluons dominate at low x $\quad xg(x) = \left(\frac{1}{x}\right)^{\lambda}$

We get for the entropy

$$S(\bar{n}) = -\sum p_n(\bar{n}) \ln p_n(\bar{n}) = \ln(\bar{n}) + (\bar{n}-1)\left(1 - \frac{1}{\bar{n}}\right)$$

$$y = \ln\left(\frac{1}{x}\right)$$

rapidity

Leading term in the low x limit

$$S(\bar{n}) = \ln xg(x)$$

In DLL approximation i.e when subsequent dipoles are strongly ordered in size and rapidity one gets:

$$S(x,Q) = \ln(xg(x,Q))$$

Krylov subspace, complexity – motivation

"The complexity of the task is defined as the minimum number of gates used to construct the circuit that accomplishes it" L. Susskind

Visvanath, Muller '63
Altman, Avdoshkin, Cao, Parker, Scaffidi '19
Balasubramanian, Caputa, Magan, Wu '22,...

State complexity

Simple reference quantum state spreads and becomes complex in Hilbert space

The key point is to expand the state or the operator in the minimal basis that supports its unitary evolution.

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} |\Psi_n\rangle$$

$$|\Psi_n\rangle \equiv \{|\Psi_0\rangle, H|\Psi_0\rangle, ..., H^n|\Psi_0\rangle, ...\}$$

Projection of a high-dimensional problem onto a lower-dimensional Krylov subspace.

Used by Krylov to understand how to efficiently diagonalize matrices.

Used as a tool in condensed matter for small Hilbert spaces.

Reviews: Baiguera et.al 2503.10753
Camargo et. al 2405.09628

Krylov subspace, complexity, entropy

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} |\Psi_n\rangle$$

$$|\Psi_n\rangle \equiv \{|\Psi_0\rangle, H|\Psi_0\rangle, ..., H^n|\Psi_0\rangle, ...\}$$

n consequtive application of Hamiltonian

Gram-Schmidt orthogonalization procedure.
Construct vector K_2 by subtracting the previous two vectors, vector K_3 by subtracting the previous 3 vectors, and so forth

Lanczos algorithm
n + 1 is determined by n and n – 1.
Low memory requirements Visvanath, Muller '63

In the Krylov basis

$$H_{nm} := \langle K_n|\hat{H}|K_m\rangle = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \cdots \\ b_1 & a_2 & b_2 & 0 & \cdots \\ 0 & b_2 & a_3 & b_3 & \cdots \\ 0 & 0 & b_3 & a_4 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle = \sum_n \phi_n(t)|K_n\rangle$$

can be at most linear in n

$$i\partial_t\phi_n(t) = a_n\phi_n(t) + b_n\phi_{n-1}(t) + b_{n+1}\phi_{n+1}(t)$$

$$p_n(t) = |\phi_n(t)|^2$$

probability amplitudes for each vector

Balasubramanian, Caputa, Magan, Wu '22

$$\mathcal{C}_K(t) = \langle n\rangle = \sum_n n\, p_n(t)$$

$$S_K(t) = -\sum_n p_n(t)\log p_n(t)$$

Krylov basis and dipole evolution

$$i\partial_t \phi_n(t) = a_n \phi_n(t) + b_n \phi_{n-1}(t) + b_{n+1}\phi_{n+1}(t)$$

The equation above can be solved for $a_n = 0 \quad b_n = \alpha n$

P. Caputa, K. Kutak, 2404.07657

boost type Hamiltonian SL(2,R) generators

$$|\psi(t)\rangle = e^{-i\alpha(L_1+L_{-1})t}\,|0\rangle \otimes |0\rangle$$

vacuum of un-resolved part of proton

vacuum of observed part of proton

One can calculate density matrix and reduced density matrix to get

$$\rho(t) = \sum_{n=0}^{\infty} p_n(t)\,|n\rangle\,\langle n|$$

$$\mathcal{C}_K(t) = \sum_{n=0}^{\infty} n|\phi_n(t)|^2$$

After expressing it in terms rapidity and probabilities rapidity variable one gets

$$\mathcal{C}_K(Y) = e^{\alpha Y} - 1$$

$$p_n(Y) = \frac{\Gamma(2h+n)}{n!\Gamma(2h)}(e^{-\alpha Y})^{2h}\left(1-e^{-\alpha Y}\right)^n$$

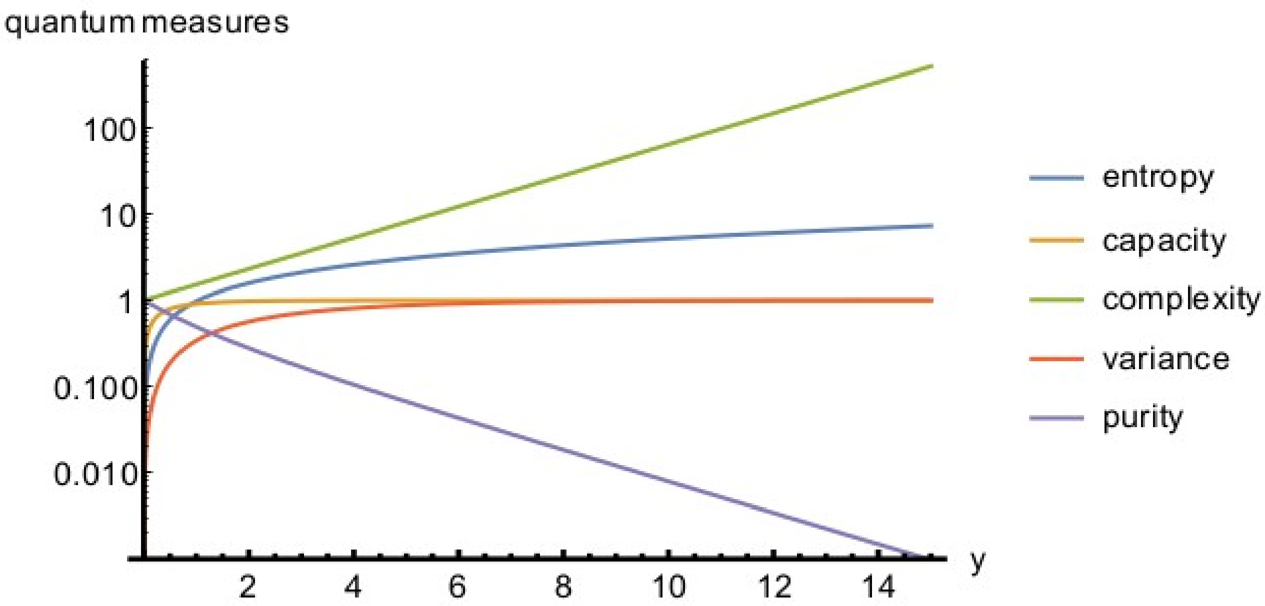
$$\mathcal{C}_K = xg(x)$$

This holds also If one accounts for transverse scales

$$\partial_Y p_n(Y) = \alpha n\, p_{n-1}(Y) - \alpha(n+1)\, p_n(Y)$$

QI measures and dipole equations

P. Caputa, K. Kutak, 2404.07657



$$S_K(t) = -\sum_n p_n(t) \log p_n(t)$$

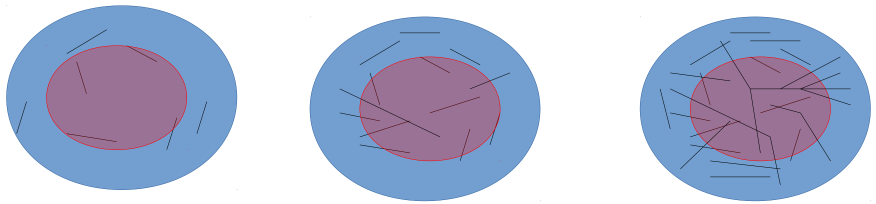
$$C_E = \lim_{m \to 1} m^2 \partial_m^2 [(1-m) S_K^{(m)}]$$

$$\mathcal{C}_K(t) = \langle n \rangle = \sum_n n\, p_n(t)$$

$$\delta_K^2 = \frac{\langle n^2 \rangle - \langle n \rangle^2}{\langle n \rangle^2}$$

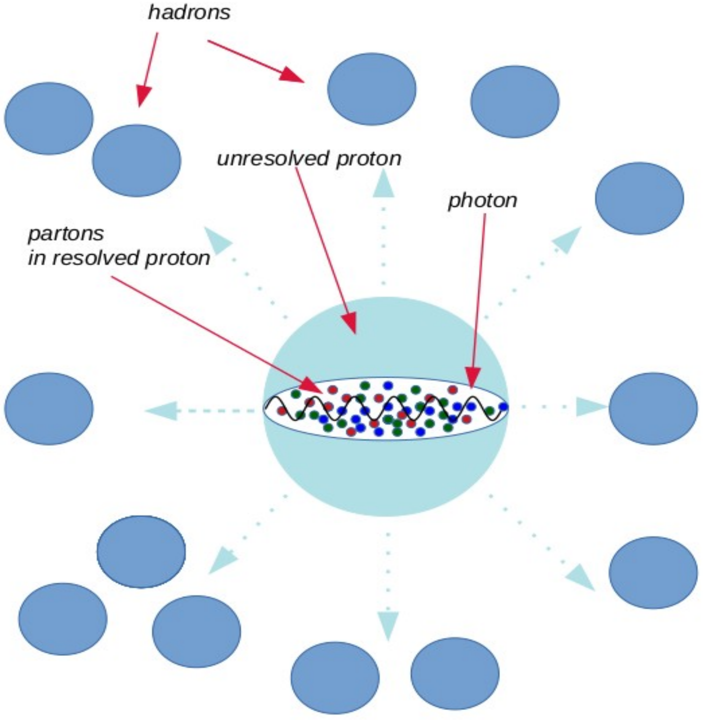
$$\gamma_K = \sum_n p_n^2(t)$$

$$\partial_Y p_n(Y) = -\lambda n p_n(Y) + \lambda(n-1) p_{n-1}(Y)$$



Entanglement entropy – calculation and measurement

For DIS at high energies, this entanglement entropy can be calculated using



$$S(x, Q^2) = \ln \left\langle n \left(\ln \frac{1}{x}, Q \right) \right\rangle$$

Calculated using parton/dipole density

Conjecture that these entropies are the same

$$S_{hadron} = \sum P(N) \ln P(N)$$

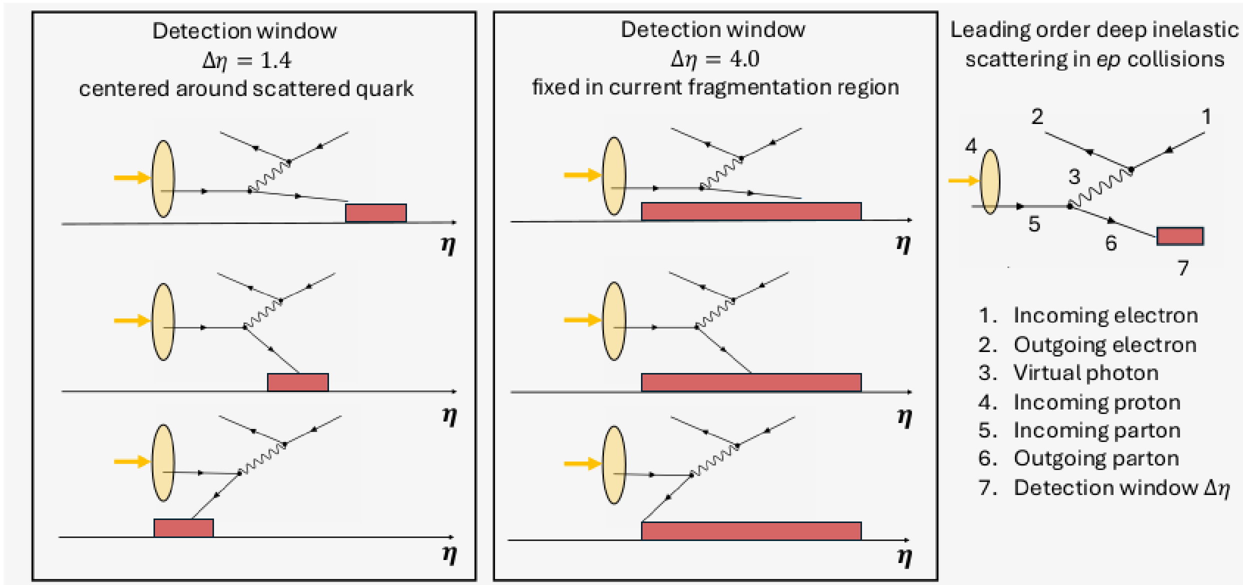
Measured by counting hadrons

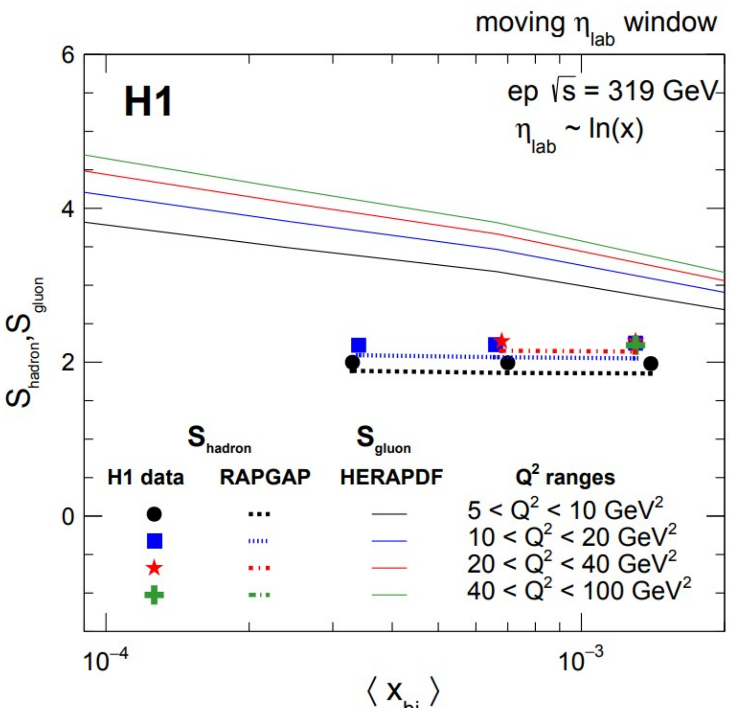
number of measured hadrons

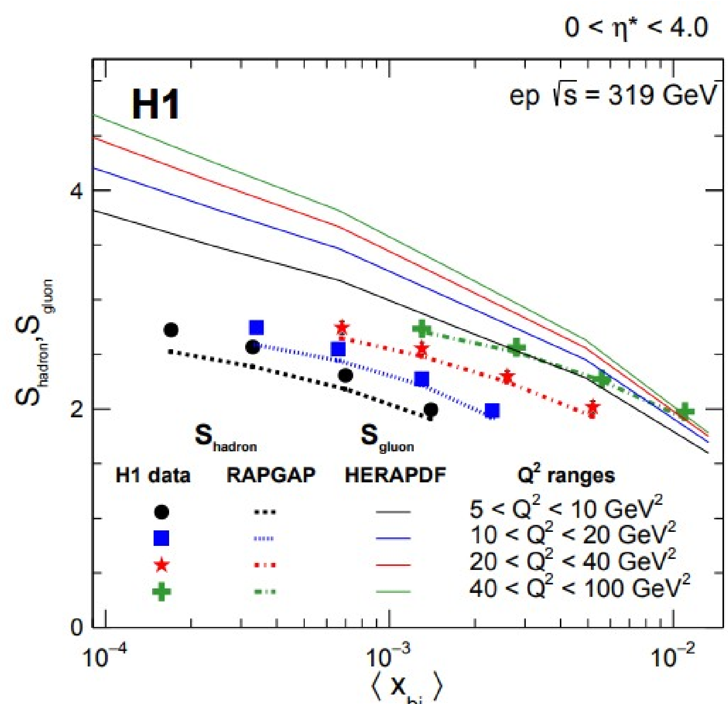
The charged particle multiplicity distribution measured in either the current fragmentation region or the target fragmentation region.

Fraction of events with charged hadron

Entropy measurements



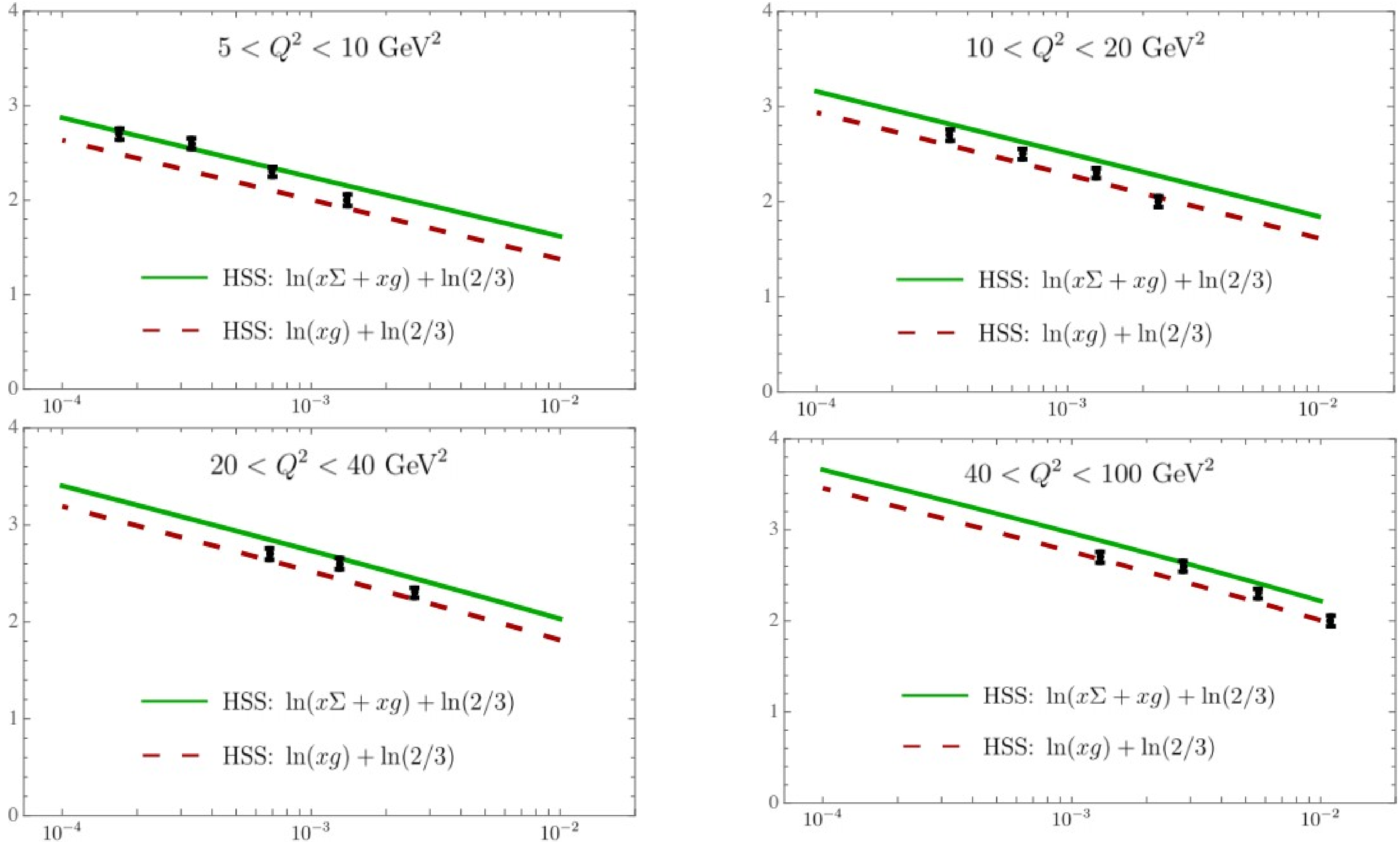




Problem with H1 escription of data...

Eur.Phys.J.C 82 (2022) 2, 111 M. Hentschinski, K. Kutak

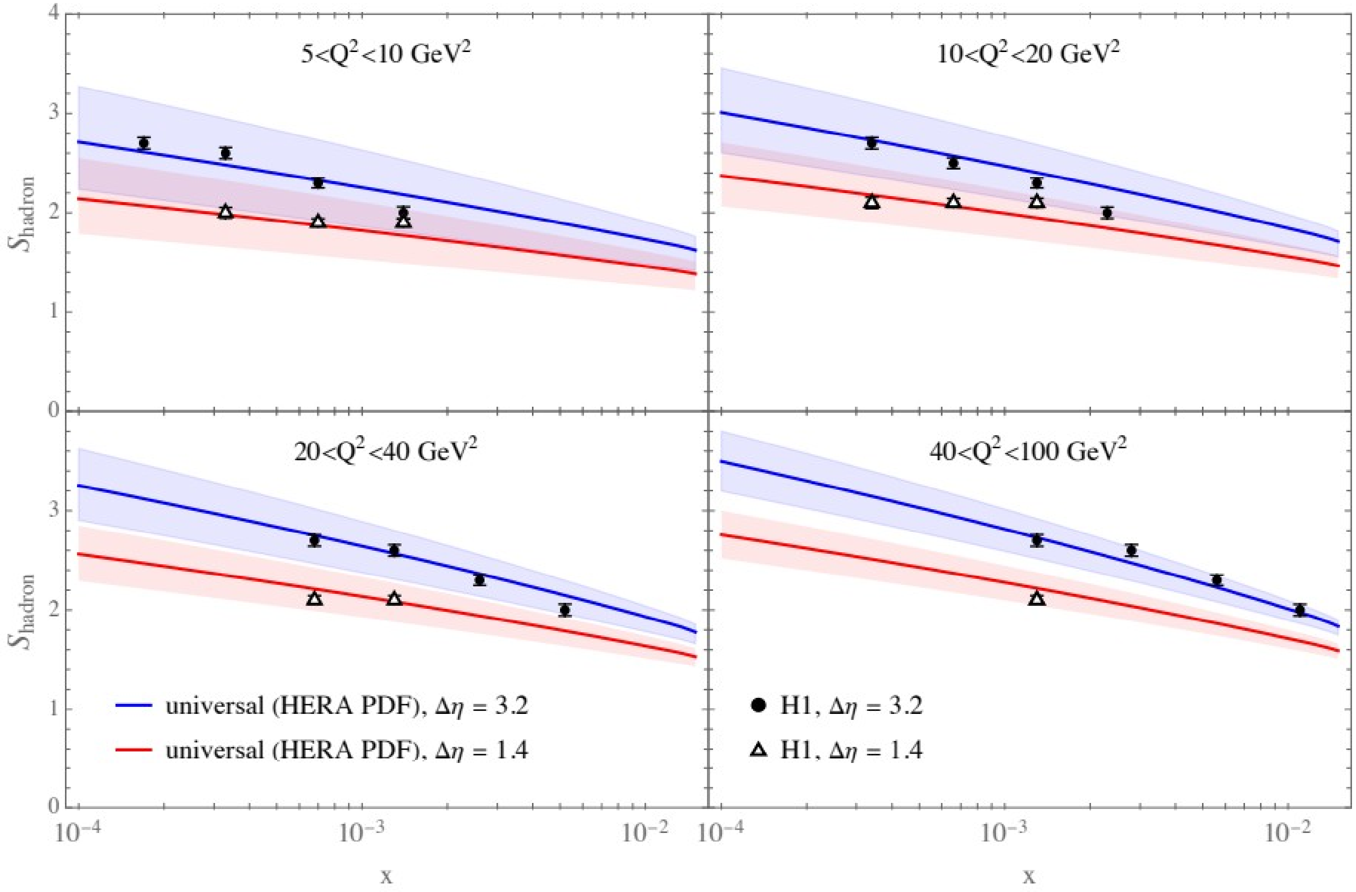
Results – fixed rapidity window



Hint that the general idea works. Gluon dominates over quarks.
One has to also take into account that only charged hadrons were measured.

Fixed and moving rapidity window description

Rept.Prog.Phys. 87 (2024) 12, 120501
M. Hentschinski, D. Kharzeev, K. Kutak, Z. Tu



blue – fixed rapidity window

red - moving rapidity window

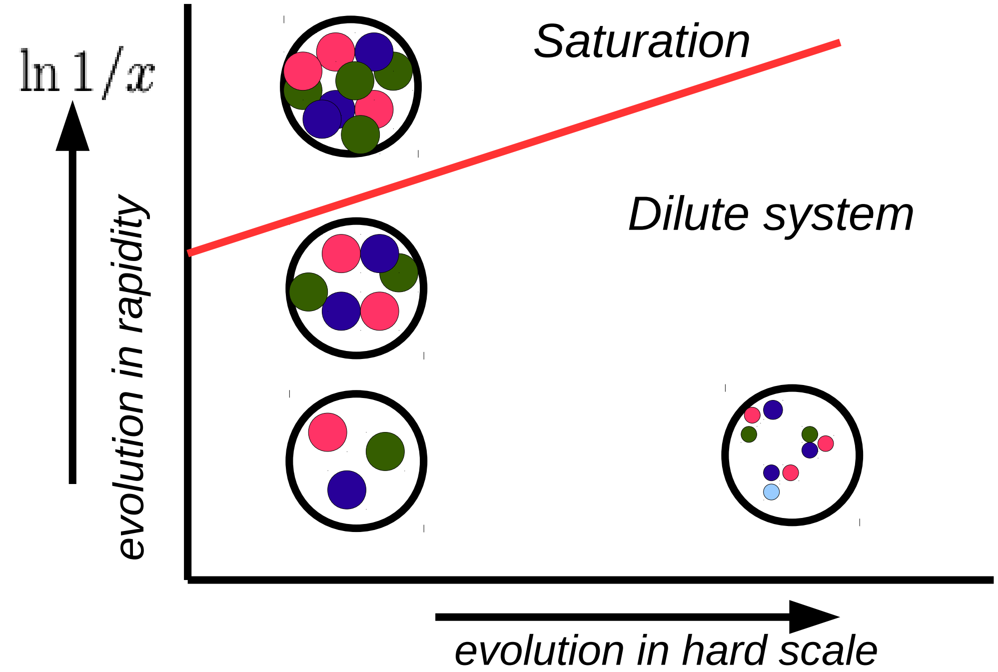
Gluon saturation – open problem of QCD

Saturation – state where number of gluons stops growing due to high occupation number. Way to fulfill unitarity requirements in high energy limit of QCD.

L.V. Gribov, E.M. Levin, M.G. Ryskin
Phys.Rept. 100 (1983) 1-150

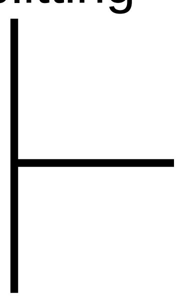
Larry D. McLerran, Raju Venugopalan
Phys.Rev. D49 (1994) 3352-3355

Phenomenological model:
Golec-Biernat, Wusthoff '99



On microscopic level it means that
gluon apart splitting recombine

splitting



Linear evolution equation
BFKL

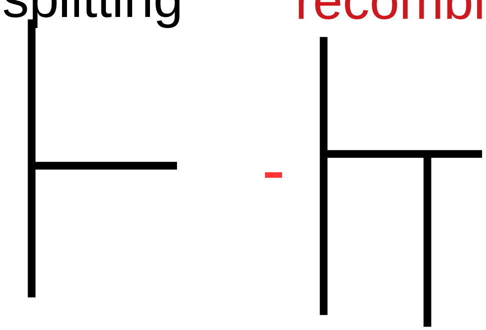
known
at NLO

Nonlinear evolution equations

BK,
Balitcky-Kovchegov,

JIMWLK
Jailian-Marian,Iancu
McLerran,Weigert,Leonidov,Kovner

splitting recombination



Bartels,
Wusthoff ‘93

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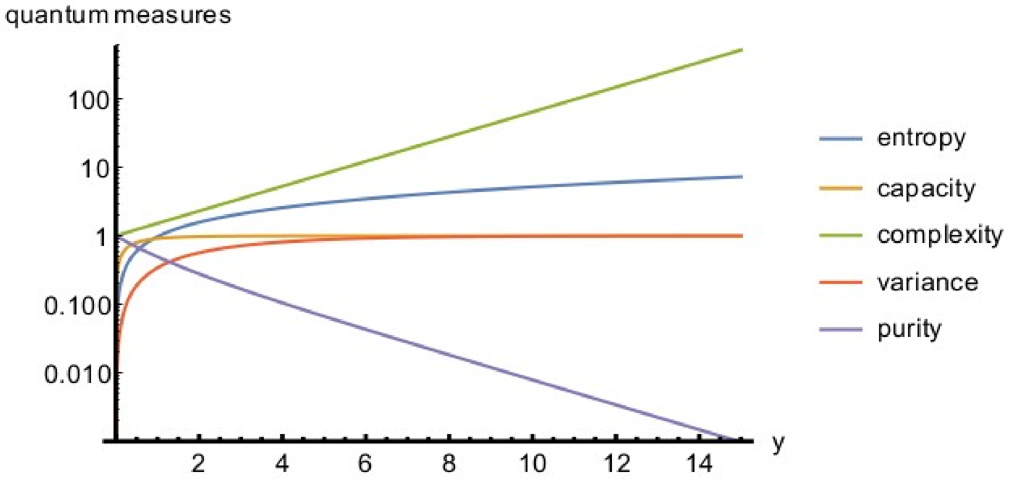
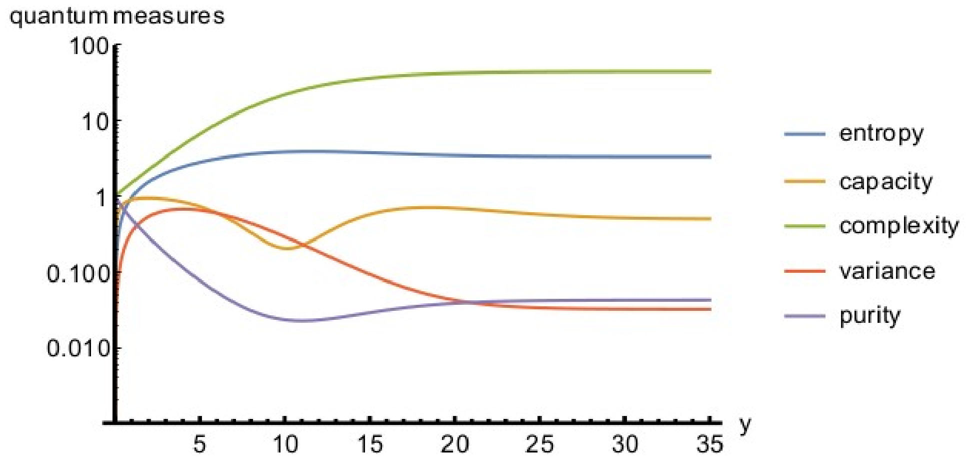
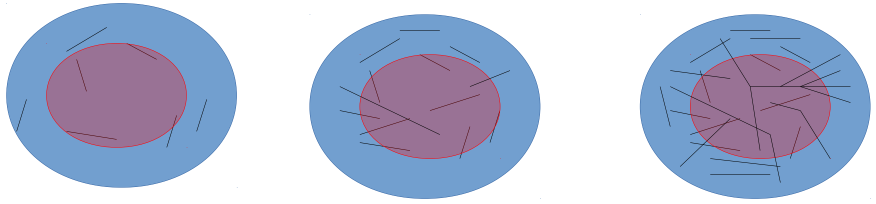




$$\partial_Y p_n(Y) = -\lambda n p_n(Y) + \lambda(n-1)p_{n-1}(Y)$$

$$\partial_Y p_n(Y) = -\lambda n p_n(Y) + \lambda(n-1)p_{n-1}(Y)$$
$$+ \beta n(n+1)p_{n+1}(Y) - \beta n(n-1)p_n(Y)$$

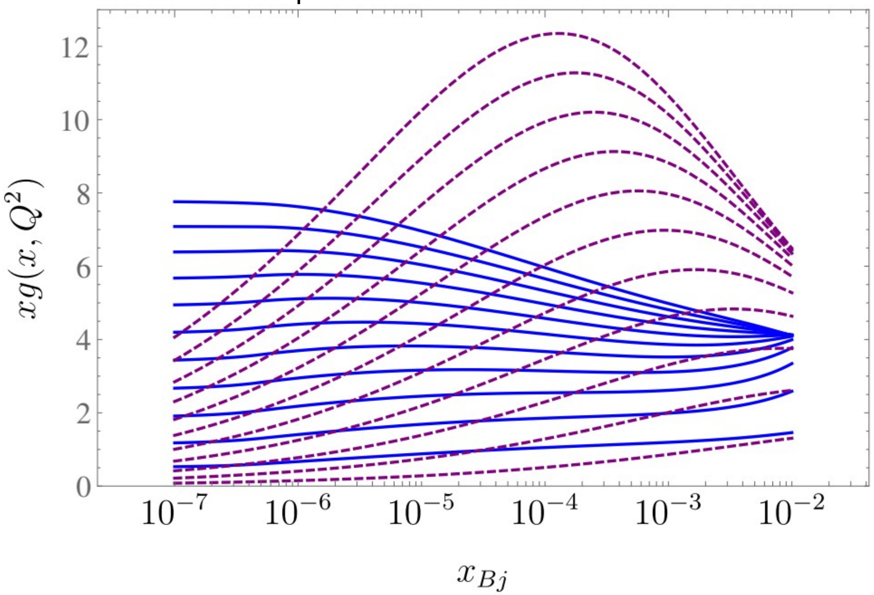
Bondarenko, Motyka, Mueller, Shoshi, Xiao ‘07

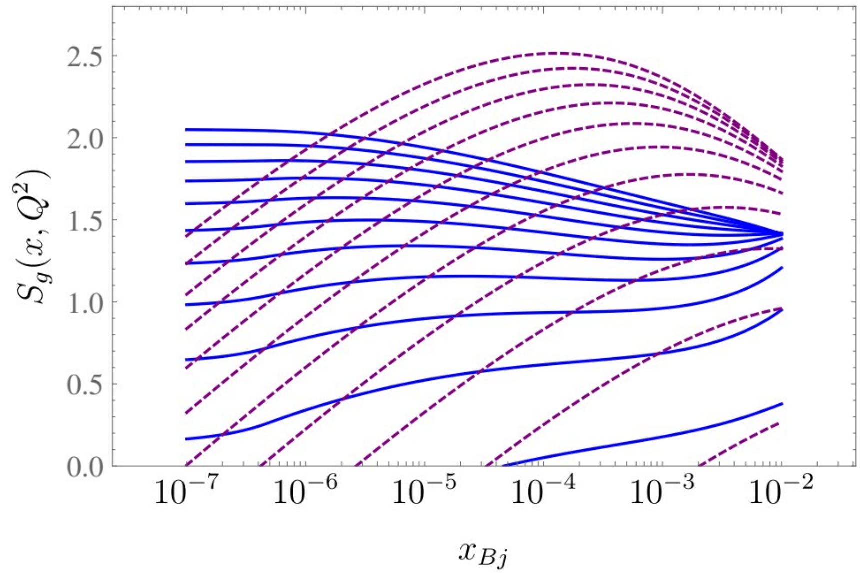


Hagiwara,Hatta,Xiao,Yuan
Phys.Rev.D 97 (2018) 9, 094029

Integrated gluon and entropy

Gluon density for various k_T
Smallest k_T at the bottom



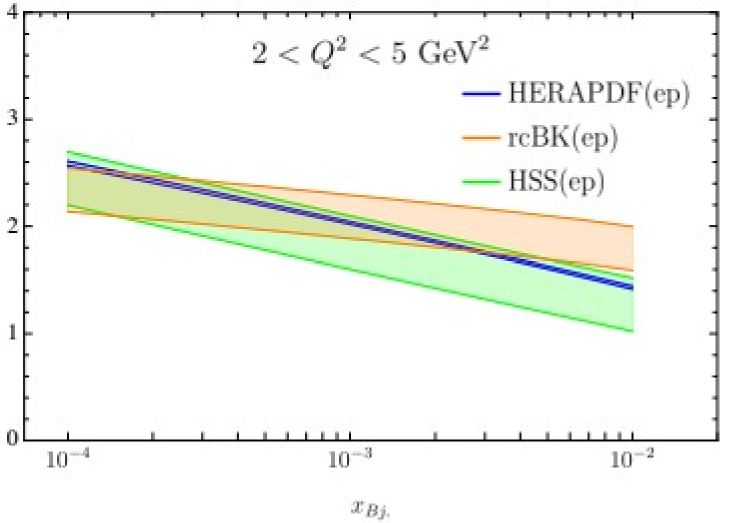


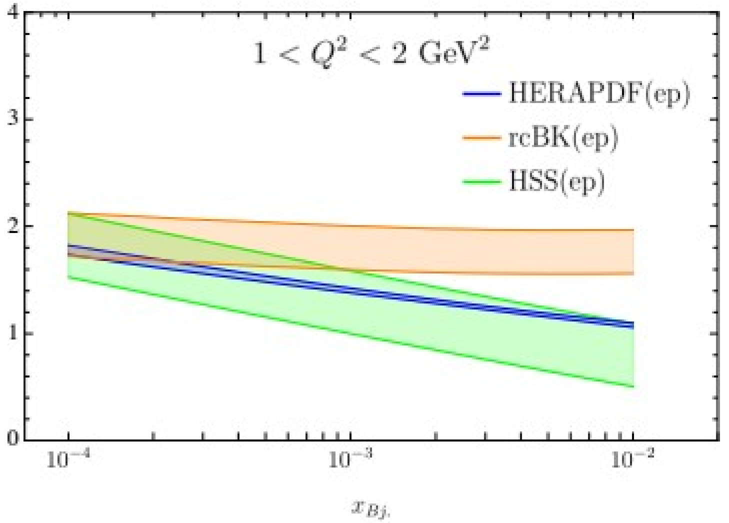
From BK equation
From GBW model

$$S(x) = \ln(xg(x))$$

Photon can not resolve proton anymore therefore the EE vanishes. But it might be that the formalism breaks down for low scales. There might be another source of entropy that keep the total entropy not vanishing → generalized second law Bekenstein

Small scales - prediction





Entropy saturates as a consequence of gluon saturation

Conclusions and outlook and comments

- Evidences for the proposal for low x maximal entanglement entropy of proton constituents.
- We related the fixed dipole size evolution equation to equation for probabilities that follow from Lanczos/Krylov construction
- Gluon density within this model corresponds to Krylov complexity
- Quantum measures - new handle to look for saturation
- Interesting in the context of Electron Ion Collider at BNL and Forward Physics Facility at CERN and forward processes in p-p and p-Pb